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The Number of Open Paths in an Oriented ρ-Percolation Model

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Abstract We study the asymptotic properties of the number of open paths of length n in an oriented ρ-percolation model. We show that this number is $e^{n\alpha(\rho)(1+o(1))}$ as $n \to \infty$. The exponent α is deterministic, it can be expressed in terms of the free energy of a polymer model, and it can be explicitly computed in some range of the parameters. Moreover, in a restricted range of the parameters, we even show that the number of such paths is $n^{-1/2}We^{n\alpha(\rho)}(1+o(1))$ for some nondegenerate random variable W. We build on connections with the model of directed polymers in random environment, and we use techniques and results developed in this context.

Keywords Oriented percolation · ρ-percolation · Directed polymers in random environment

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1 Introduction and Results

1.1 Introduction

In this paper we study the number of open paths in an oriented ρ-percolation model in dimension $1+d$, or, equivalently, the number of ρ-open path in an oriented percolation model. Consider the graph $\mathbb{Z}_+ \times \mathbb{Z}^d$, with $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$, and fix some parameter $p \in (0, 1)$. To each site of this graph except the origin, assign a variable taking value 1 with probability p and 0 with probability $1-p$, independently of the other sites. An oriented (sometimes also called *semi-oriented*) path of length n is a sequence $(0, x_0), (1, x_1), (2, x_2), \ldots, (n, x_n)$, where $x_0 = 0$ and x_i, x_{i+1} are neighbors in $\mathbb{Z}^d$, $i = 0, \ldots, n-1$: viewing the first coordinate as time, one can think of such path as a path of the d-dimensional simple random walk. Fix another parameter $\rho \in [0, 1]$; the concept of ρ-percolation was introduced by Menshikov and Zuev in [20], as the occurrence of an infinite length path with asymptotic density of 1s larger or equal to ρ. As in classical percolation, the probability of this event is subject to a dichotomy [20] according to p larger or smaller than some critical threshold, which was later studied by Kesten and Su [16] in the asymptotics of large dimension.

In the present paper, we discuss paths of finite length n, in the limit $n \to \infty$. An oriented path of length n is called ρ-open, if the proportion of 1s in it is at least ρ. From standard percolation theory it is known that for large p there are 1-open oriented paths with nonvanishing probability, and from [20] that for any p one can find ρ larger than p such that, almost surely, there are ρ-open oriented paths for large n. However, the question of how many such paths of length n can be found in a typical situation, was still unaddressed in the literature. When finishing this manuscript, we have learned of the related work [17].

In this paper, we prove that the number of different ρ-open paths of length n behaves like $e^{n\alpha(\rho)(1+o(1))}$, where the exponent $\alpha(\rho)$ is deterministic and, of course, also depends on p and d. We prove that the function $\alpha(\cdot)$ is the negative convex conjugate of the free energy of *directed polymers in random environment*. This model has attracted a lot of interest in recent years, leading to a better—although very incomplete—understanding. We will extensively use the current knowledge of thermodynamics of the polymer model, and the reader is referred to [6] for a recent survey. This will allow us to obtain, when $d \geq 3$, the explicit expression for $\alpha(\rho)$ in a certain range of values for ρ depending on the parameters p and d. The reason for this remarkable fact is the existence of the so-called "weak-disorder region" in the polymer model, discovered in [14] and [3]: this reflects here into a parameter region where the number of paths is of the same order as its expected value.

At this point the reader may be tempted to use first and second moment methods to estimate the number of paths. The first moment is easily computed, and serves as an upper bound in complete generality. The second moment is more difficult to analyse. However, it can be checked that in large dimension and for density close to the parameter p of the Bernoulli, the ratio second-to-first-squared remains bounded in the limit of an infinitely long path. This means that, under these circumstances, the upper bound gives the right order of magnitude with a positive probability. However, (i) this method does not tell us anything on α for general parameters, (ii) it fails to keep track of the correlation between counts for different values of the density.

Our strategy will be quite different. We will study the *moment generating function* of the number of paths, which is not surprising in such a combinatorial problem. The point is that the moment generating function is simply the partition function of the directed polymer in random environment. This is a well-known object in statistical physics, its logarithmic asymptotics is well studied, and is given by the free energy. From the existence and known properties of the free energy, we will derive the existence of α and its expression in thermodynamics terms. In the course of our analysis we will prove that the free energy, a convex function of the inverse temperature, is in fact strictly convex. This property is new and interesting for the polymer model.

Moreover, in a more restricted range of values for ρ, we even obtain an equivalent for the number of paths which achieves exactly a given density of 1s. This is clearly a very sharp estimate, that we obtain by using the power of complex analysis, and convergence of the renormalized moment generating function in the sense of analytic functions. Certainly a naive moments method cannot lead to such an equivalent.

1.2 Notations and Results

Now, let us define the model formally. Let $\eta(t,x)$, $t=1,2,\ldots$, $x\in\mathbb{Z}^d$ be a sequence of independent identically distributed Bernoulli random variables, with common parameter $p\in(0,1)$, $\mathbb{P}(\eta(t,x)=1)=p=1-\mathbb{P}(\eta(t,x)=0)$. We denote by $(\Omega,\mathcal{A},\mathbb{P})$ the probability space where this sequence is defined. The vertex (t,x) is *open* if $\eta(t,x)=1$ and *closed* in the opposite case. A nearest neighbor path S in $\mathbb{Z}^d$ of length n $(1\le n\le\infty)$ is a sequence $S=(S_t;t=0,\ldots,n)$, $S_t\in\mathbb{Z}^d$, $S_0=0$, $\|S_t-S_{t-1}\|_1=1$ for $t=1,\ldots,n$. We denote by $\mathcal{P}_n$ the set of such paths S, and by $\mathcal{P}_\infty$ the set of infinite length nearest neighbor paths. For $S\in\mathcal{P}_n$, let

$$H_n(S)=\sum_{t=1}^{n}\eta(t,S_t) \tag{1}$$

be the number of open vertices along the path S.

In oriented percolation, one is concerned with the event that there exists an infinite open path S, i.e.

$$\texttt{Perc}=\{\text{there exists } S\in\mathcal{P}_\infty:\eta(t,S_t)=1 \text{ for all } t\ge 1\}.$$

It is well known [12, 13] that there exists $\vec{p}_c(d)\in(0,1)$, called the critical percolation threshold, such that

$$\mathbb{P}(\texttt{Perc})\begin{cases}>0 & \text{if } p>\vec{p}_c(d),\\ =0 & \text{if } p<\vec{p}_c(d).\end{cases} \tag{2}$$

For $\rho\in(p,1]$, Menshikov and Zuev [20] introduced ρ-percolation as the event that there exists an infinite path S with asymptotic proportion at least ρ of open sites,

$$\rho\text{-}\texttt{Perc}=\Big\{\text{there exists } S\in\mathcal{P}_\infty:\liminf_{n\to\infty}H_n(S)/n\ge\rho\Big\}.$$

They showed that there also exists a threshold $\vec{p}_c(\rho,d)$ such that (2) holds with ρ-$\texttt{Perc}$ instead of $\texttt{Perc}$ (with the probability of ρ-$\texttt{Perc}$ being equal to 1 when $p>\vec{p}_c(\rho,d)$). Very little has been proved for ρ-percolation. The asymptotics of $\vec{p}_c(\rho,d)$ for large d are obtained in [16] at first order, showing that $d^{1/\rho}\vec{p}_c(\rho,d)$ has a limit as $d\to\infty$, and that the limit is different from the analogous quantity for d-ary trees. As mentioned in this reference, the equality $\vec{p}_c(1,d)=\vec{p}_c(d)$ follows from Theorem 5 of [19].

In this paper we are interested in the number of oriented paths of length n which have exactly k open vertices ($k\in\{0,\ldots,n\}$),

$$Q_n(k)=\text{Card}\{S\in\mathcal{P}_n:H_n(S)=k\}$$

(Card A denotes the cardinality of A) and the related quantity given, for $\rho \in [0,1]$, by

$$R_n(\rho) = \begin{cases} \mathrm{Card}\{S \in \mathcal{P}_n : H_n(S) \geq n\rho\}, & \rho \geq p, \\ \mathrm{Card}\{S \in \mathcal{P}_n : H_n(S) \leq n\rho\}, & \rho < p. \end{cases}$$

Note that $Q_n(k)$, $R_n(\rho)$ are random variables, that $R_n(\rho) = \sum_{k \geq n\rho} Q_n(k)$ when $\rho \geq p$, and that $\texttt{Perc} = \bigcap_n \{Q_n(n) \geq 1\} = \bigcap_n \{R_n(1) \geq 1\}$.

In this paper we relate these quantities to the model of directed polymers in random environment. Central in this model is the (unnormalized) partition function $Z_n = Z_n(\beta, \eta)$ at inverse temperature $\beta \in \mathbb{R}$ in the environment η given by

$$Z_n = \sum_{S \in \mathcal{P}_n} \exp\{\beta H_n(S)\}.$$

By subadditive arguments one can prove that

$$\varphi(\beta) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \ln Z_n \tag{3}$$

exists in $\mathbb{R}$ ($\mathbb{E}$ is the expectation under $\mathbb{P}$), and by concentration arguments, that the event $\Omega_0(\beta)$ defined by

$$\Omega_0(\beta) = \left\{ \lim_{n \to \infty} \frac{1}{n} \ln Z_n = \varphi(\beta) \right\} \tag{4}$$

has full measure, $\mathbb{P}(\Omega_0(\beta)) = 1$, see e.g. [5]. The function φ is called the *free energy*, it is a non-decreasing and convex function of β. Its Legendre conjugate

$$\varphi^*(\rho) = \sup\{\beta\rho - \varphi(\beta); \beta \in \mathbb{R}\}, \tag{5}$$

is a convex, lower semi-continuous function from $[0,1]$ to $\mathbb{R} \cup \{+\infty\}$, such that $\varphi^*(\rho) \geq \varphi^*(p) = -\ln(2d)$ (indeed, $\varphi'(0) = p$, as it will be shown later). Legendre convex duality is better understood by taking a glance at the graphical construction, e.g. figures 2.2.1 and 2.2.2 in [9]; here, on Fig. 1 we illustrate how the functions φ and φ^* typically look in our situation. The existence of the so-called time constants,

$$\rho^+ = \lim_{n \to \infty} \max_{S \in \mathcal{P}_n} \frac{H_n(S)}{n}, \qquad \rho^- = \lim_{n \to \infty} \min_{S \in \mathcal{P}_n} \frac{H_n(S)}{n}, \qquad \mathbb{P}\text{-a.s.}$$

can be obtained by specifying a direction for the ending point S_n, which allows using subadditive arguments [15], and then summing over the possible directions. However we give here a short proof in the spirit of this work. Since $\exp\{\beta \max_{S \in \mathcal{P}_n} H_n(S)\} \leq Z_n \leq (2d)^n \exp\{\beta \max_{S \in \mathcal{P}_n} H_n(S)\}$, we have

$$\frac{1}{n\beta} \ln Z_n - \frac{1}{\beta} \ln(2d) \leq \max_{S \in \mathcal{P}_n} \frac{H_n(S)}{n} \leq \frac{1}{n\beta} \ln Z_n.$$

Taking the limits $n \to \infty$ and then $\beta \to +\infty$, we see that ρ^+ is well-defined, and is in fact equal to the slope $\lim_{\beta \to +\infty} \varphi(\beta)/\beta$ of the asymptotic direction of φ at $+\infty$. From standard properties of convex duality, the range of the derivative $(d/d\beta)(1/n) \ln Z_n(\beta)$ a.s. converges to $[\rho^-, \rho^+]$ in Hausdorff distance, and $\varphi^*(\rho) < +\infty$ if and only if $\rho \in [\rho^-, \rho^+]$. For such ρ, we have $\varphi^*(\rho) \leq 0$.

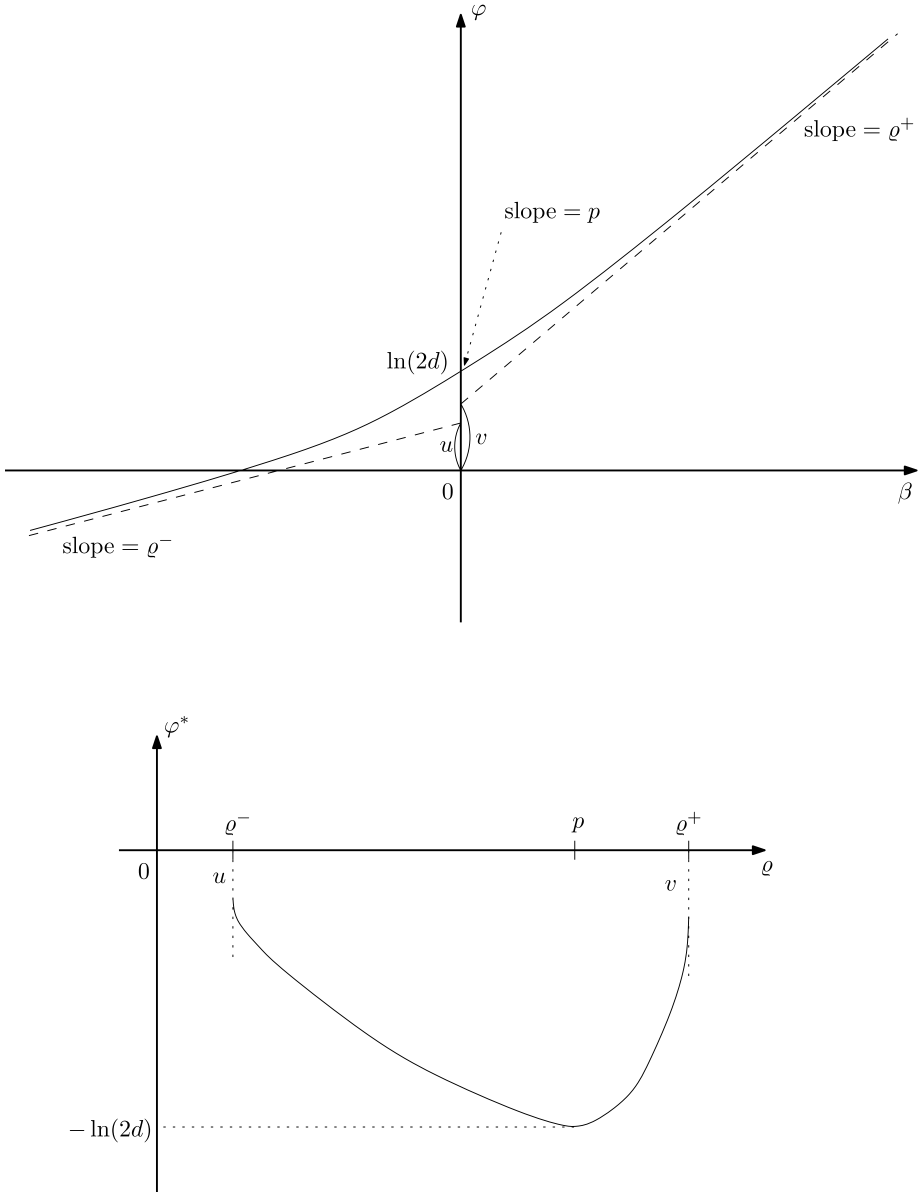

Fig. 1 The function φ and its Legendre transform φ^*

Theorem 1.1 *For all $\rho \in [0, 1]$ with $\rho \neq \rho^+, \rho^-$, the following limit*

$$\alpha(\rho) = \lim_{n\to\infty} \frac{1}{n} \ln R_n(\rho) \tag{6}$$

exists $\mathbb{P}$-a.s. (possibly assuming the value $-\infty$), and is given by

$$\alpha(\rho) = -\varphi^*(\rho).$$

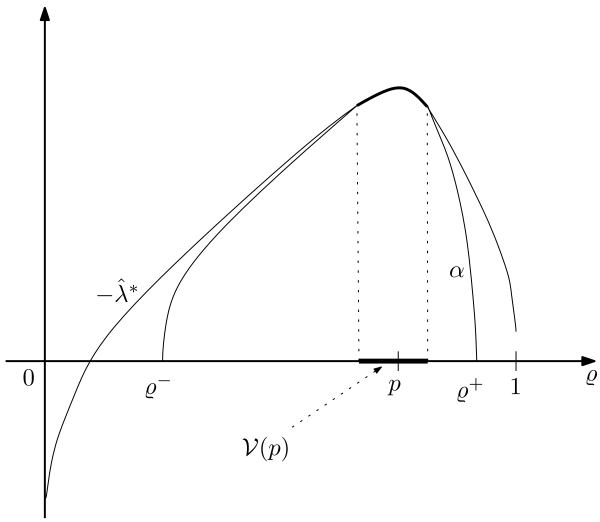

Fig. 2 Typical behaviour of the function α when $d \geq 3$

Clearly, α is concave, with values in $[0, \ln(2d)] \cup \{-\infty\}$ and $\alpha(p) = \ln(2d)$. Note that, for all $\rho \in (\rho^-, \rho^+)$ and almost every η,

$$R_n(\rho) = \exp n[\alpha(\rho) + o(1)], \quad \text{as } n \to \infty.$$

Remark 1.2 By convexity the function α is continuous on (ρ^-, ρ^+). For now, it is not clear to us whether the limit $\alpha(\rho^+-)$ should be equal to 0 in the case $p \leq \vec{p}_c(d)$. In the case $p > \vec{p}_c(d)$, it is possible to show by subadditive arguments that, conditionally on percolation, the limit $\alpha(1)$ in (6) exists and is positive, but it is not clear to us whether α is continuous at 1.

Let

$$\lambda(\beta) = \ln \mathbb{E} e^{\beta\eta(t,x)} = \ln[1 + p(e^\beta - 1)], \qquad \hat{\lambda}(\beta) = \lambda(\beta) + \ln(2d),$$

then $\mathbb{E}Z_n = \exp\{n\hat{\lambda}(\beta)\}$. A direct computation shows that the Legendre conjugate $\hat{\lambda}^*(\rho) = \sup\{\beta\rho - \hat{\lambda}(\beta); \beta \in \mathbb{R}\}$ of $\hat{\lambda}$ is equal to

$$\begin{aligned} \hat{\lambda}^*(\rho) &= -\ln(2d) + \rho \ln \frac{\rho}{p} + (1-\rho) \ln \frac{1-\rho}{1-p} \\ &= \rho \ln \frac{\rho}{2dp} + (1-\rho) \ln \frac{1-\rho}{2d(1-p)}. \end{aligned} \tag{7}$$

The function $(-\hat{\lambda}^*)$ is important for understanding the rate α. (We recall that both functions depend on p, but we do not write explicitly the dependence.) Note that these two functions coincide at $\rho = p$ and take the value $\ln(2d)$.

Theorem 1.3 *Let* $p \in (0, 1)$.

1. *We have* the annealed bound: *For all* ρ,

$$\alpha(\rho) \leq -\hat{\lambda}^*(\rho). \tag{8}$$

2. *The function* $\alpha(\rho) + \hat{\lambda}^*(\rho)$ *is nonincreasing for* $\rho \in [p, \rho^+)$ *and is nondecreasing for* $\rho \in (\rho^-, p]$.
3. *The set*

$$\mathcal{V}(p) = \{\rho \in (0, 1) : \alpha(\rho) = -\hat{\lambda}^*(\rho)\} \tag{9}$$

   is an interval containing p (*here, "interval" is understood in broad sense, i.e., it can reduce to the single point* $\{p\}$).
4. *In dimension* $d = 1$, $\mathcal{V}(p) = \{p\}$, *i.e. the inequality in* (8) *is strict for all* $\rho \neq p$.
5. *In dimension* $d \geq 3$, $\mathcal{V}(p)$ *contains a neighborhood of* p.
6. *Let* $d \geq 3$, *and* π_d *be the probability for the* d*-dimensional simple random walk to ever return to the starting point. When* $p > \pi_d$, *then* $[p, 1) \subset \mathcal{V}(p)$, *so that the equality holds in* (8) *for all* $\rho \in [p, 1)$. *Similarly, when* $p < 1 - \pi_d$, *then* $(0, p] \subset \mathcal{V}(p)$, *so that the equality holds for all* $\rho \in (0, p]$.
7. *In dimension* $d \geq 2$, *if* $p < (1/2d)$, *then* $\sup \mathcal{V}(p) < 1$. *Similarly, if* $p > 1 - (1/2d)$, *we have* $\inf \mathcal{V}(p) > 0$.

See Fig. 2 for the typical shape of the function α.

Remark 1.4

(i) The annealed bound comes from the first-moment method, and most of the results stating that the equality $\alpha(\rho) = -\hat{\lambda}^*(\rho)$ holds, are derived from the second-moment method.
(ii) By transience of the random walk in dimension $d \geq 3$, we have $\pi_d < 1$. In fact, $\pi_3 = 0.3404\ldots > \pi_4 > \pi_5 \ldots$ [24, page 103]. In particular, for $p \in (\pi_d, 1 - \pi_d)$, we have $\alpha(\rho) = -\hat{\lambda}^*(\rho)$ for all $\rho \in (0, 1)$.

The following property of the free energy φ of the directed polymer is interesting and seems to be new.

Theorem 1.5 *The function* φ *is strictly convex on* $\mathbb{R}$, *and the functions* φ^* *and* α *are differentiable in the interior of their domains.*

We will obtain much sharper results for large dimension and ρ's not too far from p. The reason is that the partition function Z_n behaves smoothly as $n \nearrow \infty$. The almost-sure limit

$$W_\infty(\beta) = \lim_{n \to \infty} Z_n(\beta) e^{-n\hat{\lambda}(\beta)}$$

exists for all β, since the sequence is a positive $(\mathcal{G}_n)_n$-martingale, where $\mathcal{G}_n = \sigma\{\eta(t, x); t \leq n, x \in \mathbb{Z}^d\}$. So, let us now concentrate on the case of large dimension, $d \geq 3$. When β belongs to some neighborhood of the origin (known as the weak disorder region), the limit W_∞ is strictly positive a.s. In a smaller neighborhood of the origin, the limit can be expressed as a (random) perturbation series in $\mathcal{L}^2$ [23]. Moreover, the convergence holds in much stronger sense, namely, in the sense of analytic functions [8]. We will use strong tools from complex analysis, as it is classically done to obtain limit theorems for sums of random variables [21].

Theorem 1.6 *Assume* $d \geq 3$. *There exist a neighborhood* U_3 *of* p *in* $\mathbb{R}$ *and an event* Ω_2 *with full probability such that for every sequence* k_n *with* $k_n/n \to \rho \in U_3$ *and all* $\eta \in \Omega_2$,

$$Q_n(k_n) = \sqrt{\frac{-\alpha''(\rho)}{2\pi n}} W_\infty(\beta(\rho)) \exp\left\{n\alpha\left(\frac{k_n}{n}\right)\right\}(1 + o(1))$$

where $o(1)$ tends to 0 as $n \to \infty$, and $\beta(\rho) = \ln \frac{(1-p)\rho}{p(1-\rho)}$. The neighborhood U_3 is contained in $\mathcal{V}(p)$, hence we have $\alpha = -\hat{\lambda}^*$ with $\hat{\lambda}^*$ given by (7).

We note that the leading order is deterministic, but the prefactor is random (as W_∞), depending on the particular realization of the Bernoulli field. This theorem is a corollary of a more refined result (Theorem 3.2), which can be found in Sect. 3. This will be proved by complex analysis arguments, considering the Fourier transform of H_n under some (polymer) measure. Fourier methods are quite strong, they are used in a different spirit in [2] to obtain sharp results on the polymer path itself for small β. Uniform convergence of analytic martingales have been already proved in the (related) model of supercritical branching random walks [1], leading to sharp controls of the local growth of the population. The disadvantage is that we have to restrict the parameter domain. It would be tempting to use only real variable techniques as in the Ornstein-Zernike theory for the Bernoulli bond percolation [4], but we take another, shorter route.

Remark 1.7 The model is also interesting with real-valued $\eta(t, x)$ with general distribution. This is motivated by first-passage time percolation. Our results at the exponential order remain valid for variables with exponential moments. For the case of the Gaussian law, we mention the recent work [18] on the so-called REM conjecture: it is proved that the local statistics of $(H_n(S); S \in \mathcal{P}_n)$ approach that of a Poisson point process, provided that one focuses on values distant from the mean $\mathbb{E}H_n$ by at most $o(n^{1-\varepsilon})$.

We can interpret our last result in this spirit. In our case, $(H_n(S); S \in \mathcal{P}_n)$ spreads on the lattice, and natural local statistics of the energy levels are the ratios $Q_n(k_n)/\mathbb{E}Q_n(k_n)$. For $d \geq 3$ and $k_n \sim n\rho \in U_3$,

$$Q_n(k_n)/\mathbb{E}Q_n(k_n) \simeq W_\infty(\beta(\rho))$$

since $\mathbb{E}W_\infty(\beta) = 1$. We emphasize that here the energy level k_n is of order n (far from the bulk), and that the limit is not universal but depends on the lattice and the law of the environment η.

2 Logarithmic Asymptotics

2.1 Proof of Theorem 1.1

We start by introducing some probability measures.

Let P be the law of the simple random walk on $\mathbb{Z}^d$ starting from 0, i.e. the probability measure on the space $\mathcal{P}_\infty$ of infinite paths making the increments $S_t - S_{t-1}$ independent and uniformly distributed on the set of $2d$ neighbors of $0 \in \mathbb{Z}^d$. Observe that the restriction of P to paths of length n is the normalized counting measure on $\mathcal{P}_n$, and so the partition function takes now the familiar form (E_P is the expectation with respect to P)

$$Z_n = (2d)^n E_P[\exp\{\beta H_n(S)\}]. \tag{10}$$

The law $\nu_n = \nu_n^\eta$ of $(1/n)H_n$ under P, given by $\nu_n(\{\rho\}) := P(H_n(S) = n\rho)$, is such that

$$\nu_n(\{\rho\}) = \frac{Q_n(n\rho)}{(2d)^n} \quad \text{if } n\rho \in \{0, 1, \dots, n\}. \tag{11}$$

We extend ν_n to a probability measure on $\mathbb{R}$ that we still denote by ν_n, $\nu_n(A) = \sum_{\rho\in A, n\rho\in\{0,1,\dots,n\}} \nu_n(\{\rho\})$, $A\subset\mathbb{R}$.

All what we need to obtain the proof of Theorem 1.1, is to prove that ν_n obeys an almost sure large deviation principle, see Proposition 2.1 below. Recall first the event $\Omega_0(\beta)$ from (4), and define the event $\Omega_0 = \bigcap_{\beta\in\mathbb{Q}} \Omega_0(\beta)$. Then, we have $\mathbb{P}(\Omega_0) = 1$, and on this event the convergence (4) holds for any real number β by convexity and monotonicity.

Proposition 2.1 *The function*

$$I(\rho) = \ln(2d) + \varphi^*(\rho) \in [0, \ln(2d)] \cup \{+\infty\}$$

is lower semi-continuous and convex on $[0, 1]$. *Moreover, for all* $\eta \in \Omega_0$ *the sequence* $(\nu_n, n \geq 1)$ *obeys a large deviation principle with rate function* I. *That is,*

(i) *for any closed* $F \subset [0, 1]$, *we have*

$$\limsup_{n\to\infty} n^{-1} \ln \nu_n(F) \leq -\inf_{\rho\in F} I(\rho),$$

(ii) *for any open* (*in the induced topology on* $[0, 1]$) $G \subset [0, 1]$, *we have*

$$\liminf_{n\to\infty} n^{-1} \ln \nu_n(G) \geq -\inf_{\rho\in G} I(\rho).$$

Now, we first finish the proof of Theorem 1.1, and then prove the above proposition.

Proof of Theorem 1.1 Assume that $\rho \in [p, \rho^+)$ and $\eta \in \Omega_0$. Applying (i) of Proposition 2.1 with $F = [\rho, 1]$ and using (11) together with the fact that $\rho \geq p$, we see that the limit in (6) is not larger than $\ln(2d) - I(\rho) = \alpha(\rho)$. Applying (ii) of Proposition 2.1 with $G = (\rho + \varepsilon, 1]$ ($\varepsilon > 0$) and using the fact that $\rho \geq p$, we see that the limit is at least $\alpha(\rho + \varepsilon)$. Since $\rho < \rho^+$, this quantity tends to $\alpha(\rho)$ as $\varepsilon \searrow 0$. This proves (6) for $\rho \in [p, \rho^+)$. The case $\rho \in (\rho^-, p)$ is completely similar. Finally, when $\rho > \rho^+$ (the case $\rho < \rho^-$ is similar) we have $I(\rho) = \infty$ and then $R_n(\rho) = 0$ for large n, proving (6) in this case. □

Proof of Proposition 2.1 The properties of I are clear from the definition.

Fix $\eta \in \Omega_0$. In view of (4) and (10), the Laplace transforms of $\nu_n(\cdot) = P((1/n)H_n = \cdot)$ have logarithmic asymptotics:

$$\lim_{n\to\infty} \frac{1}{n} \ln E_P(\exp\{\beta H_n(S)\}) = \varphi(\beta) - \ln(2d)$$

for all real β. From the Gärtner-Ellis theorem (Theorem 2.3.6 in [9]), the full statement (i) in Proposition 2.1 follows, and we obtain for open $G \subset [0, 1]$ that

$$\liminf_{n\to\infty} \frac{1}{n} \ln \nu_n(G) \geq -\inf\{I(\rho); \rho \in G \cap \mathcal{E}\}, \tag{12}$$

where

$$\mathcal{E} = \left\{\rho \in [0, 1] : \exists\beta\ \forall r \neq \rho, \beta\rho - \varphi^*(\rho) > \beta r - \varphi^*(r)\right\}$$

is the set of exposed points of φ^* from (5). Its complement is the set of all points ρ such that φ^* is linear in a non-trivial interval containing ρ. We will improve (12) into (ii) of

Proposition 2.1 with a subadditivity argument. We start by showing that φ is differentiable at 0 with that $\varphi'(0) = p$. Indeed, using Jensen inequality twice, we have

$$\frac{1}{n}\ln E_P[e^{\mathbb{E}\beta(H_n-np)}] \le \mathbb{E}\frac{1}{n}\ln E_P[e^{\beta(H_n-np)}] \le \frac{1}{n}\ln \mathbb{E}E_P[e^{\beta(H_n-np)}].$$

Computing the extreme terms and taking the limit $n\to\infty$ for the middle one, we get

$$0 \le \varphi(\beta) - \beta p \le \lambda(\beta) - \beta p,$$

which shows that $\varphi'(0) = p$ since $\lambda'(0) = p$. This implies that $p \in \mathcal{E}$ and that $\mathcal{E}$ is a neighborhood of p. Let $\rho \in (\rho^-, \rho^+) \cap G$ be a non-exposed point of φ^*. For definiteness, we assume $\rho > p$. Let

$$\rho_1 = \sup\{\rho' \in \mathcal{E}; \rho' < \rho\}, \qquad \rho_2 = \inf\{\rho' \in \mathcal{E}; \rho' > \rho\}.$$

Recall that φ is strictly convex by Theorem 1.5—that we will prove below independently. This implies that the function φ^* cannot have a linear piece that goes up to ρ^+, cf. Fig. 1. Then, $p < \rho_1 < \rho < \rho_2 < \rho^+$, and $\rho_1, \rho_2 \in \mathcal{E}$. Let $\gamma \in (0,1)$ such that $\rho = \gamma\rho_1 + (1-\gamma)\rho_2$. Since the interval (ρ_1, ρ_2) consists of non-exposed points, we have $I(\rho) = \gamma I(\rho_1) + (1-\gamma)I(\rho_2)$. Since G is open and contains ρ, we can find $\varepsilon > 0$ and $k, \ell \in \mathbb{N}^*$ such that

$$|u-\rho_1| < \varepsilon, \quad |v-\rho_2| < \varepsilon \quad \Longrightarrow \quad \frac{ku+\ell v}{k+\ell} \in G^\varepsilon$$

with G^ε the set of $r \in G$ at distance at least ε from the outside of G. The key fact is

$$\begin{aligned}
&\mathrm{Card}\left\{S \in \mathcal{P}_{n(k+\ell)} : \frac{H_{n(k+\ell)}(S)}{n(k+\ell)} \in G^\varepsilon\right\}\\
&\quad\ge \sum_{x\in\mathbb{Z}^d} \mathrm{Card}\left\{S \in \mathcal{P}_{n(k+\ell)} : \frac{H_{nk}(S)}{nk} \in (\rho_1-\varepsilon, \rho_1+\varepsilon), S_{nk} = x\right\}\\
&\qquad\times \mathrm{Card}\left\{S \in \mathcal{P}_{n(k+\ell)} : S_{nk} = x, \frac{H_{n(k+\ell)}(S) - H_{nk}(S)}{n\ell} \in (\rho_2-\varepsilon, \rho_2+\varepsilon)\right\}\\
&\quad\ge \mathrm{Card}\left\{S \in \mathcal{P}_{n(k+\ell)} : \frac{H_{nk}(S)}{nk} \in (\rho_1-\varepsilon, \rho_1+\varepsilon)\right\}\\
&\qquad\times \min_{\|x\|_1\le nk} \mathrm{Card}\left\{S \in \mathcal{P}_{n(k+\ell)} : S_{nk} = x,\right.\\
&\qquad\qquad\qquad\left.\frac{H_{n(k+\ell)}(S) - H_{nk}(S)}{n\ell} \in (\rho_2-\varepsilon, \rho_2+\varepsilon)\right\}\\
&\quad= \mathrm{Card}\left\{S \in \mathcal{P}_{nk} : \frac{H_{nk}(S)}{nk} \in (\rho_1-\varepsilon, \rho_1+\varepsilon)\right\}\\
&\qquad\times \min_{\|x\|_1\le nk} \mathrm{Card}\left\{S \in \mathcal{P}_{n\ell} : \frac{H_{n\ell}^{(nk,x)}(S)}{n\ell} \in (\rho_2-\varepsilon, \rho_2+\varepsilon)\right\}
\end{aligned}$$

with $H_{n\ell}^{(nk,x)}(S) = \sum_{t=1}^n \eta(t+nk, S_t+x)$ the Hamiltonian in the time-space shifted environment. Similarly, we denote by $\nu_{n\ell}^{(nk,x)}$ the measure $\nu_{n\ell}^{(nk,x)}(\cdot) = P(H_{n\ell}^{(nk,x)} \in \cdot)$. The above display implies that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n(k+\ell)} \ln \nu_{n(k+\ell)}(G^{\varepsilon}) \\
&\geq \frac{k}{k+\ell} \liminf_{n\to\infty} \frac{1}{nk} \ln \nu_{nk}((\rho_1-\varepsilon, \rho_1+\varepsilon)) \\
&\quad + \frac{\ell}{k+\ell} \liminf_{n\to\infty} \frac{1}{n\ell} \min_{\|x\|_1 \leq nk} \ln \nu_{n\ell}^{(nk,x)}((\rho_2-\varepsilon, \rho_2+\varepsilon)).
\end{aligned}$$

It is straightforward to check that

$$\liminf_{n\to\infty} \frac{1}{n(k+\ell)} \ln \nu_{n(k+\ell)}(G^{\varepsilon}) \leq \liminf_{n\to\infty} \frac{1}{n} \ln \nu_n(G),$$

and it is not difficult to see that

$$\liminf_{n\to\infty} \frac{1}{n\ell} \min_{\|x\|_1 \leq nk} \ln \nu_{n\ell}^{(nk,x)}((\rho_2-\varepsilon, \rho_2+\varepsilon)) \geq -I(\rho_2), \quad \mathbb{P}\text{-a.s.} \tag{13}$$

We postpone the proof of (13) for the moment. Hence, the key inequality implies

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \ln \nu_n(G) \geq -\frac{k}{k+\ell} I(\rho_1+\varepsilon) - \frac{\ell}{k+\ell} I(\rho_2+\varepsilon), \\
&\liminf_{n\to\infty} \frac{1}{n} \ln \nu_n(G) \geq -[\gamma I(\rho_1) + (1-\gamma) I(\rho_2)] = -I(\rho),
\end{aligned}$$

letting $\varepsilon \searrow 0$ and $k/(k+\ell) \to \gamma$. This yields statement (ii) in Proposition 2.1.

Now, let us prove (13). By a standard concentration inequality (e.g., Theorem 4.2 in [7]), we have

$$\mathbb{P}(|\ln Z_n - \mathbb{E} \ln Z_n| \geq u) \leq 2 \exp\left\{-\frac{u^2}{4\beta^2 n}\right\}.$$

Therefore we have, $\mathbb{P}$-a.s. as $n \to \infty$,

$$\max_{\|x\|_1 \leq m \leq n} \left| \frac{1}{n} \ln Z_n^{(m,x)}(\beta) - \varphi(\beta) \right| \to 0, \quad \beta \in \mathbb{R},$$

with $Z_n^{(m,x)}$ the partition function associated to $H_n^{(m,x)}$. Since ρ_2 is an exposed point for φ^*, (13) follows from the Gärtner-Ellis theorem. □

Let us comment on the above proof. We could improve (12) into the full lower bound (ii) in Proposition 2.1 with a subadditivity argument, implying convexity of the rate function. If we knew that $(\rho^-, \rho^+) \subset \mathcal{E}$—or, equivalently, that φ is differentiable,—we could directly conclude without this extra argument. We tried to prove it, but we could not. We state it as a conjecture:

Conjecture 2.2 The function φ is everywhere differentiable.

2.2 Proof of Theorem 1.3

1. By Jensen inequality,
$$\mathbb{E}\ln Z_n \leq n\hat{\lambda}(\beta).$$
Then, $\varphi(\beta) \leq \hat{\lambda}(\beta)$, which implies $\varphi^*(\rho) \geq \hat{\lambda}^*(\rho)$ from the definition of Legendre transform. The inequality now follows from $\alpha \leq -\varphi^*$.
2. Set $\varphi_n(\beta) = n^{-1}\mathbb{E}\ln Z_n(\beta)$. From Theorem 1.1 in [8] we have
$$\hat{\lambda}'(\beta) \geq \varphi_n'(\beta)$$
for all $\beta \geq 0$. Hence, for $\rho \geq p$, the reciprocal functions are such that
$$(\hat{\lambda}')^{-1}(\rho) \leq (\varphi_n')^{-1}(\rho).$$
Since $(\hat{\lambda}')^{-1} = (\hat{\lambda}^*)'$ and $(\varphi_n')^{-1} = (\varphi_n^*)'$, we have
$$(\hat{\lambda}^*)'(\rho) \leq (\varphi^*)'(\rho)$$
for all $\rho \geq p$ where φ^* is differentiable. Since $\alpha = -\varphi^*$ for $\rho \neq \rho^+$, this proves the first half of the desired statement. The other half is similar.
3. From Theorem 1.1 in [8] it is known that the set
$$\mathcal{W}(p) = \{\beta \in \mathbb{R} : \varphi(\beta) = \hat{\lambda}(\beta)\}$$
is an interval containing 0. Let $\beta \in \mathcal{W}(p)$, and $\rho = \lambda'(\beta) = \hat{\lambda}'(\beta)$. From Theorem 2.3 (a) in [5] it is known that $\beta \in \mathcal{W}(p)$ implies $\varphi^*(\rho) \leq 0$. Then, the supremum defining $\hat{\lambda}^*(\rho)$ is achieved at β, which implies the first equality in
$$-\hat{\lambda}^*(\rho) = -[\beta\rho - \hat{\lambda}(\beta)] = -[\beta\rho - \varphi(\beta)] = -\varphi^*(\rho) = \alpha(\rho),$$
where the second equality holds for $\beta \in \mathcal{W}(p)$, the third one because of $\varphi'(\beta) = \hat{\lambda}'(\beta) = \rho$, and the last one because $\varphi^*(\rho) \leq 0$.

   Let now $\beta \notin \mathcal{W}(p)$, and $\rho = \lambda'(\beta)$. Then,
$$-\hat{\lambda}^*(\rho) = -[\beta\rho - \hat{\lambda}(\beta)] > -[\beta\rho - \varphi(\beta)] \geq -\varphi^*(\rho) \geq \alpha(\rho).$$
Observe that λ' is a diffeomorphism from $\mathbb{R}$ to $(0, 1)$. From this we can identify the set $\mathcal{V}(p)$ defined by (9),
$$\mathcal{V}(p) = \{\lambda'(\beta); \beta \in \mathcal{W}(p)\}, \tag{14}$$
which is an interval containing p.
4. When $d = 1$, it is known that $\mathcal{W}(p) = \{0\}$, see Theorem 1.1 in [8]. Hence, $\mathcal{V}(p)$ reduces to $\{p\}$.
5. When $d \geq 3$, from celebrated results of Imbrie and Spencer [14], Bolthausen [3], it is known that $\mathcal{W}(p)$ contains a neighborhood of 0. In view of (14), $\mathcal{V}(p)$ is in its turn a neighborhood of p.
6. This is a consequence of [6, example 2.1.1], which shows for instance that, if $p > \pi_d$, then $\mathcal{W}(p) \supset \mathbb{R}^+$. Indeed, in view of (14), this implies that $\mathcal{V}(p)$ contains $[p, 1)$, and α is still equal to $-\hat{\lambda}^*$ at $\rho = 1$ by upper semi-continuity of both functions. The case of $p < 1 - \pi_d$ is similar.
7. This is a consequence of [6, example 2.2.1], which shows for instance that, if $p < (1/2d)$, then $\mathcal{W}(p)$ is bounded from above. The other case is similar.

2.3 Strict Convexity of the Free Energy

The aim of this section is to prove Theorem 1.5. We start with a variance estimate analogous to that for Gibbs field in [10].

Lemma 2.3 *For any compact set $K \subset \mathbb{R}$, there exists a positive constant $C = C_K$ such that*

$$\mathbb{E}(\ln Z_n)''(\beta) \geq Cn, \quad \beta \in K.$$

Proof The polymer measure at inverse temperature β with environment η is the random probability measure $\mu_n = \mu_n^\beta$ on the path space defined by

$$\mu_n(\{S\}) = Z_n^{-1} \exp\{\beta H_n(S)\}, \quad S = (S_1, \dots, S_n) \in \mathcal{P}_n. \tag{15}$$

For simplicity we write $\mu_n(S_1, \dots, S_n)$ for $\mu_n(\{(S_1, \dots, S_n)\})$. The polymer measure is Markovian (but time-inhomogeneous), and

$$(\ln Z_n)''(\beta) = \mathrm{Var}_{\mu_n}(H_n).$$

Let Σ_t be the t-coordinate mapping on $\mathcal{P}_n$ given by $\Sigma_t(S) = S_t$, and regard it as a random variable.

Define

$$\mathcal{I}(x, y) = \{z \in \mathbb{Z}^d : \|x - z\|_1 = \|z - y\|_1 = 1\}, \quad x, y \in \mathbb{Z}^d \tag{16}$$

the set of lattice points which are next to both x and y. The set $\mathcal{I}(x, y)$ is empty except if y can be reached in two steps by the simple random walk from x; in this case its cardinality is equal to $2d, 2$ or 1 according to $y = x$, $\|y - x\|_\infty = 1$ or $\|y - x\|_\infty = 2$. The Markov property implies that, under μ_{2n}, $\Sigma_1, \Sigma_3, \dots, \Sigma_{2n-1}$ are independent conditionally on $\Sigma^e := (\Sigma_2, \Sigma_4, \dots, \Sigma_{2n})$, and the law of Σ_{2t-1} given Σ^e only depends on $\Sigma_{2t-2}, \Sigma_{2t}$, and has support $\mathcal{I}(\Sigma_{2t-2}, \Sigma_{2t})$.

From the variance decomposition under conditioning, we have

$$\begin{aligned}
\mathrm{Var}_{\mu_{2n}}(H_{2n}) &= E_{\mu_{2n}} \mathrm{Var}_{\mu_{2n}}(H_{2n} \mid \Sigma^e) + \mathrm{Var}_{\mu_{2n}}(E_{\mu_{2n}}[H_{2n} \mid \Sigma^e]) \\
&\geq E_{\mu_{2n}} \mathrm{Var}_{\mu_{2n}}(H_{2n} \mid \Sigma^e) \\
&= E_{\mu_{2n}} \mathrm{Var}_{\mu_{2n}}\left(\sum_{t=1}^{n} \eta(2t-1, \Sigma_{2t-1}) \mid \Sigma^e\right) \\
&= \sum_{t=1}^{n} E_{\mu_{2n}} \mathrm{Var}_{\mu_{2n}}(\eta(2t-1, \Sigma_{2t-1}) \mid \Sigma^e)
\end{aligned}$$

where $E[\,\cdot \mid \Sigma^e]$, $\mathrm{Var}(\,\cdot \mid \Sigma^e)$ denote conditional expectation and conditional variance. To obtain the last equality we used the conditional independence. Define the event

$$M(\eta, t, y, z) = \left\{\mathrm{Card}\left\{\eta(t, x); x \in \mathcal{I}(y, z)\right\} = 2\right\}.$$

The reason for introducing $M(\eta, t, y, z)$ is that on this event, a path S conditioned on $S_{t-1} = y$, $S_{t+1} = z$, has the option to pick up a $\eta(t, S_t)$ value that can be either 0 or 1, bringing therefore some amount of randomness. This event plays a key role here, as well as in the proof of Lemma 3.3 below. Note for further purpose that

$$\mathbb{P}\big(M(\eta,t,y,z)\big) = 1-\big(p^{\mathrm{Card}\,\mathcal{I}(y,z)}+(1-p)^{\mathrm{Card}\,\mathcal{I}(y,z)}\big) =: q(y-z). \tag{17}$$

The key observation is, for all $t \le n$ and $\beta \in K$,

$$\mathrm{Var}_{\mu_{2n}}(\eta(2t-1,\Sigma_{2t-1}) \mid \Sigma^e) \ge C\mathbf{1}\{M(\eta,2t-1,\Sigma_{2t-2},\Sigma_{2t})\}, \tag{18}$$

where the constant C depends only on K and the dimension d. Indeed, on the event $M(\eta,2t-1,S_{2t-2},S_{2t})$, the variable $\eta(2t-1,\Sigma_{2t-1})$ brings some fluctuation under the conditional law: it takes values 0 and 1 with probability uniformly bounded away from 0 provided β remains in the compact set. Hence,

$$\begin{aligned}
\mathbb{E}\mathrm{Var}_{\mu_{2n}}(H_{2n}) &\ge C\mathbb{E}\sum_{t=1}^{n}\mu_{2n}[M(\eta,2t-1,\Sigma_{2t-2},\Sigma_{2t})]\\
&= C\mathbb{E}\sum_{t=1}^{n}\sum_{x,y\in\mathbb{Z}^d}\mu_{2n}(\Sigma_{2t-2}=x,\Sigma_{2t}=y)\mathbf{1}\{M(\eta,2t-1,x,y)\}.
\end{aligned}$$

For $1\le i\le n$, let $\tilde{\mu}_n^{(i)}$ be the polymer measure in the environment $\tilde{\eta}(t,x)=\eta(t,x)$ if $t\ne i$, $\tilde{\eta}(i,x)=0$ for all x. Obviously,

$$C^-\tilde{\mu}_n^{(i)}(S) \le \mu_n(S) \le C^+\tilde{\mu}_n^{(i)}(S), \qquad S\in\mathcal{P}_n,$$

with positive finite C^-, C^+ not depending on $n,\eta,\beta\in K$. Then, with $C'=CC^-$,

$$\begin{aligned}
&\mathbb{E}\mathrm{Var}_{\mu_{2n}}(H_{2n})\\
&\quad\ge C'\mathbb{E}\sum_{t=1}^{n}\sum_{x,y\in\mathbb{Z}^d}\tilde{\mu}_{2n}^{(2t-1)}(\Sigma_{2t-2}=x,\Sigma_{2t}=y)\mathbf{1}\{M(\eta,2t-1,x,y)\}\\
&\quad= C'\mathbb{E}\sum_{t=1}^{n}\sum_{x,y\in\mathbb{Z}^d}\tilde{\mu}_{2n}^{(2t-1)}(\Sigma_{2t-2}=x,\Sigma_{2t}=y)\mathbb{P}(M(\eta,2t-1,x,y))\\
&\quad\ge 2C'p(1-p)\mathbb{E}\sum_{t=1}^{n}\sum_{x,y\in\mathbb{Z}^d}\tilde{\mu}_{2n}^{(2t-1)}(\Sigma_{2t-2}=x,\Sigma_{2t}=y)\mathbf{1}\{\|x-y\|_\infty\le 1\}\\
&\quad\ge C'p(1-p)\mathbb{E}\sum_{t=2}^{2n}\sum_{x,y\in\mathbb{Z}^d}\tilde{\mu}_{2n}^{(2t-1)}(\Sigma_{t-2}=x,\Sigma_{t}=y)\mathbf{1}\{\|x-y\|_\infty\le 1\},
\end{aligned}$$

since we can repeat the same procedure, but conditioning on the path at odd times. Finally, with $\Delta S_t := S_t - S_{t-1}$ and $C''=C'C^-p(1-p)$, we have for all $\beta\in K,\varepsilon>0$,

$$\begin{aligned}
\mathbb{E}\mathrm{Var}_{\mu_{2n}}(H_{2n}) &\ge C''\mathbb{E}E_{\mu_{2n}}\sum_{t=2}^{2n}\mathbf{1}\{\Delta\Sigma_t\ne\Delta\Sigma_{t-1}\}\\
&\ge nC''\varepsilon\times\mathbb{E}\mu_{2n}(A_{n,\varepsilon}),
\end{aligned} \tag{19}$$

where

$$A_{n,\varepsilon} = \left\{ S \in \mathcal{P}_n : \sum_{t=2}^{2n} \mathbf{1}\{\Delta S_t \neq \Delta S_{t-1}\} \geq n\varepsilon \right\}.$$

It is easy to see that the complement

$$A_{n,\varepsilon}^c = \left\{ \sum_{t=2}^{2n} \mathbf{1}\{\Delta S_t = \Delta S_{t-1}\} > n(2-\varepsilon) \right\}$$

of this set has cardinality smaller than $\exp\{n\delta(\varepsilon)\}$, with $\delta(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$. We bound

$$\mathbb{P}(\max\{H_{2n}(S);\, S \in A_{n,\varepsilon}^c\} \geq 2n\rho) \leq e^{n\delta(\varepsilon)} \times \mathrm{Prob}(\mathcal{B}(2n, p) \geq 2n\rho),$$

with $\mathcal{B}(2n, p)$ a binomial random variable. It follows that there exists some $\rho(\varepsilon)$ with $\rho(\varepsilon) \searrow p$ as $\varepsilon \searrow 0$ such that the left-hand side is less than $\exp\{-n\delta(\varepsilon)^{1/2}\}$. For all η such that $\max\{H_{2n}(S);\, S \in A_{n,\varepsilon}^c\} \leq 2n\rho(\varepsilon)$, we have the estimate

$$\begin{aligned} \mu_{2n}(A_{n,\varepsilon}^c) &\leq \exp\{2n[\beta\rho(\varepsilon) - \varphi(\beta) + \delta(\varepsilon) + o(1)]\} \\ &\leq \exp\left\{2n[\varphi^*(\rho(\varepsilon)) + \delta(\varepsilon) + o(1)]\right\} \end{aligned}$$

with $o(1) \to$ as $n \to \infty$. But, as $\varepsilon \searrow 0$,

$$\varphi^*(\rho(\varepsilon)) + \delta(\varepsilon) \to \varphi^*(p) = -\ln(2d) < 0.$$

By continuity we can choose $\varepsilon > 0$ such that $\varphi^*(\rho(\varepsilon)) + \delta(\varepsilon) \leq (-1/2)\ln(2d)$, and $\mathbb{E}\mu_{2n}(A_{n,\varepsilon}) \to 1$ as $n \to \infty$. Finally, from (19) we obtain the desired result for even n. The same computations apply to μ_{2n+1}, yielding a similar bound. This concludes the proof of Lemma 2.3. □

Proof of Theorem 1.5 It follows from Lemma 2.3 that, for $\beta, \beta' \in K$,

$$\varphi(\beta') \geq \varphi(\beta) + (\beta' - \beta)\varphi_r'(\beta) + \frac{C_K}{2}(\beta' - \beta)^2, \quad \beta \leq \beta',$$

with φ_r' the right-derivative, and a similar statement for $\beta' \leq \beta$. Indeed, this inequality holds for $(1/n)\mathbb{E}\ln Z_n$ instead of φ, and we can pass to the limit $n \to \infty$. This yields the strict convexity of φ. By a classical property of Legendre duality, it implies the differentiability of φ^*. □

3 Sharp Asymptotics

Assume $d \geq 3$. Let U_0 be the open set in the complex plane given by $U_0 = \{\beta \in \mathbb{C} : |\operatorname{Im}\beta| < \pi\}$. Then, U_0 is a neighborhood of the real axis, and $\lambda(\beta) = \log\mathbb{E}[\exp\{\beta\eta(t,x)\}]$ is an analytic function on U_0. Define, for $n \geq 0$ and $\beta \in U_0$,

$$W_n(\beta) = E_P\left[\exp\left(\beta\sum_{t=1}^{n}\eta(t, S_t) - n\lambda(\beta)\right)\right]. \tag{20}$$

Then, for all $\beta \in U_0$, the sequence $(W_n(\beta), n \geq 0)$ is a $(\mathcal{G}_n)_n$-martingale with complex values, where $\mathcal{G}_n = \sigma\{\eta(t,x); t \leq n, x \in \mathbb{Z}^d\}$. At the same time, for each n and η, $W_n(\beta)$ is an analytic function of $\beta \in U_0$.

Define the real subset

$$U_1 = \{\beta \in \mathbb{R} \,:\, \lambda(2\beta) - 2\lambda(\beta) < -\ln \pi_d\}, \tag{21}$$

which is an open interval (β_1^-, β_1^+) containing 0 $(-\infty \leq \beta_1^- < 0 < \beta_1^+ \leq +\infty)$. The following is established in [8]:

Proposition 3.1 *Define U_2 to be the connected component of the set*

$$\big\{\beta \in U_0 \,:\, \lambda(2\,\mathrm{Re}\,\beta) - 2\,\mathrm{Re}\,\lambda(\beta) < -\ln \pi_d\big\}$$

which contains the origin. Then, U_2 is a complex neighborhood of U_1. Furthermore, there exists an event Ω_1 with $\mathbb{P}(\Omega_1) = 1$ such that,

$$W_n(\beta) \to W_\infty(\beta) \quad \textit{as } n \to \infty, \textit{ for all } \eta \in \Omega_1, \beta \in U_2,$$

where the convergence is locally uniform. In particular, the limit $W_\infty(\beta)$ is holomorphic in U_2, and all derivatives of W_n converge locally uniformly to the corresponding ones of W_∞. Finally, $W_\infty(\beta) > 0$ for all $\beta \in U_1$, $\mathbb{P}$-a.s.

For the sake of completeness we repeat the proof here.

Proof of Proposition 3.1 Since $\overline{(e^z)} = e^{\overline{z}}$ and $\overline{\mathbb{E}[f]} = \mathbb{E}[\overline{f}]$, we have $\overline{\lambda(\beta)} = \lambda(\overline{\beta})$, and

$$\begin{aligned}
\mathbb{E}\big[|W_n(\beta)|^2\big] &= \mathbb{E}\big[E_P[\exp\{\beta H_n(S) - n\lambda(\beta)\}]E_P[\exp\{\overline{\beta} H_n(\tilde{S}) - n\overline{\lambda(\beta)}\}]\big] \\
&= E_{P^{\otimes 2}}\big[\mathbb{E}\big[\exp\{\beta H_n(S) + \overline{\beta} H_n(\tilde{S}) - 2n\,\mathrm{Re}\,\lambda(\beta)\}\big]\big] \\
&= E_{P^{\otimes 2}}\left[\exp\left\{[\lambda(2\,\mathrm{Re}\,\beta) - 2\,\mathrm{Re}\,\lambda(\beta)]\sum_{t=1}^{n} \mathbf{1}\{S_t = \tilde{S}_t\}\right\}\right] \\
&\leq E_{P^{\otimes 2}}\left[\exp\left\{[\lambda(2\,\mathrm{Re}\,\beta) - 2\,\mathrm{Re}\,\lambda(\beta)]\sum_{t=1}^{\infty} \mathbf{1}\{S_t = \tilde{S}_t\}\right\}\right] \\
&< \infty
\end{aligned} \tag{22}$$

if $\beta \in U_2$. Indeed, the random variable $\sum_{t=1}^{\infty} \mathbf{1}\{S_t = \tilde{S}_t\}$ (which is the number of meetings between two independent d-dimensional simple random walks) is geometrically distributed with parameter π_d.

For any real $\beta \in U_2$, the positive martingale $W_n(\beta)$ is bounded in L^2, hence it converges almost surely and in L^2-norm to a non-negative limit $W_\infty(\beta)$. Moreover, the event $\{W_\infty(\beta) = 0\}$ is a tail event, so it has probability 0 or 1. Since $\mathbb{E}W_\infty(\beta) = 1$, we have necessarily $W_\infty(\beta) > 0$, $\mathbb{P}$-a.s.

We need a stronger convergence result. Fix a point $\beta \in U_2$ and a radius $r > 0$ such that the closed disk $D(\beta, r) \subset U_2$. Choosing $R > r$ such that $D(\beta, R) \subset U_2$, we obtain by Cauchy's integral formula for all $\beta' \in D(\beta, r)$,

$$W_n(\beta') = \frac{1}{2i\pi}\int_{\partial D(\beta,R)} \frac{W_n(z)}{z - \beta'}dz = \int_0^1 \frac{W_n(\beta + Re^{2i\pi u})Re^{2i\pi u}}{(\beta + Re^{2i\pi u}) - \beta'}du,$$

hence

$$X_n := \sup\{|W_n(\beta')|;\ \beta' \in D(\beta, r)\} \le R \int_0^1 \frac{|W_n(\beta + Re^{2i\pi u})|}{R - r} du.$$

Letting $C = (R/(R-r))^2$, we obtain by the Schwarz inequality

$$\begin{aligned}(\mathbb{E}[X_n])^2 &\le C\mathbb{E}\left[\int_0^1 |W_n(\beta + Re^{2i\pi u})|^2 du\right]\\ &\le C \sup\{\mathbb{E}[|W_n(\beta'')|^2];\ n \ge 1, \beta'' \in D(\beta, R)\}\\ &< \infty\end{aligned}$$

in view of (22). Notice now that X_n, a supremum of positive submartingales, is itself a positive submartingale. Since $\sup \mathbb{E}[X_n] < \infty$, X_n converges $\mathbb{P}$-a.s. to a finite limit X_∞. Finally,

$$\sup\{|W_n(\beta')|;\ \beta' \in D(\beta, r), n \ge 1\} < \infty \quad \mathbb{P}\text{-a.s.},$$

and W_n is uniformly bounded on compact subsets of U_2 on a set of environments of full probability. On this set, $(W_n, n \ge 0)$ is a normal sequence [22] which has a unique limit on the real axis: since U_2 is connected, the full sequence converges to some limit W_∞, which is holomorphic on U_2, and, as mentioned above, positive on the real axis. □

We do not know that $W_\infty(\beta) \ne 0$ for general $\beta \in U_2$, only for $\beta \in U_1$. Therefore, for all $\eta \in \Omega_1$, we fix another complex neighborhood U_3 of U_1, included in U_2 and depending on η, such that W_∞ and W_n (for n large) belongs to $\mathbb{C} \setminus \mathbb{R}_-$. Recall that

$$Z_n(\beta) = W_n(\beta) \exp\{n\hat{\lambda}(\beta)\} \tag{23}$$

by definition.

It is sometimes convenient to consider, for real β, the β-tilted law

$$\nu_{n,\beta}(k) = Z_n(\beta)^{-1} e^{\beta k} Q_n(k), \quad k \in \{0, 1, \dots, n\},$$

which is a probability measure on the integers $0, 1, \dots, n$. Its mean is equal to $(d/d\beta)$ $\ln Z_n(\beta)$, and its variance is

$$D_{n,\beta} = \frac{d^2}{d\beta^2} \ln Z_n(\beta). \tag{24}$$

These quantities depend also on η, and $D_{n,\beta} > 0$ as soon as the Bernoulli configuration $(\eta(t, x), t \le n, \|x\|_1 \le n, \|x\|_1 = n \bmod 2)$ is not identically 0 or 1 on each "hyperplane" $t = k$, $k = 1, \dots, n$. This happens eventually with probability 1, so we will not worry about degeneracy of the variance $D_{n,\beta}$. By positivity of the variance, for all u in the range of $(d/d\beta) \ln Z_n(\cdot)$ there exists unique $\beta = \beta_n(u) \in \mathbb{R}$ such that

$$\frac{d}{d\beta} \ln Z_n(\beta_n(u)) = u. \tag{25}$$

Observe that the function β_n is itself random. Define for $\beta \in \mathbb{R}$, $k \in \mathbb{N}$,

$$I_n(k) = \sup\{\beta k - \ln Z_n(\beta);\ \beta \in \mathbb{R}\} - n \ln(2d). \tag{26}$$

(We will see in the proof of Theorem 1.6 below, that $I_n(k) \sim nI(k/n)$ with I as in Proposition 2.1.) For k in the range of $(d/d\beta)\ln Z_n(\cdot)$, we have

$$I_n(k) = \beta_n(k)k - \ln Z_n(\beta_n(k)) - n\ln(2d). \tag{27}$$

Recall (β_1^-, β_1^+) defined in (21).

Theorem 3.2 *There exist an event Ω_2 with $\mathbb{P}(\Omega_2) = 1$ and a real neighborhood U_4 of* 0, *$U_4 \subset (\beta_1^-, \beta_1^+)$, with the following property. Let $k_n \in \{0, 1, \dots, n\}$ be a sequence such that $\beta_n(k_n)$ remains in a compact subset K of U_4, and let $\hat{D}_n = D_{n,\beta_n(k_n)}$. Then, for all $\eta \in \Omega_2$,*

$$Q_n(k_n) = \frac{1}{\sqrt{2\pi \hat{D}_n}} \exp\{-I_n(k_n) + n\ln(2d)\} \times (1 + o(1)),$$

where $o(1) \to 0$ as $n \to \infty$.

Proof of Theorem 3.2 Suppose that β is a real number. Note that the Fourier transform of the tilted measure is

$$\sum_{k=0}^{n} e^{iku} \nu_{n,\beta}(k) = \frac{Z_n(\beta + iu)}{Z_n(\beta)}.$$

From the usual inversion formula for Fourier series we have

$$Q_n(k_n) = Z_n(\beta)e^{-\beta k_n} \times \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Z_n(\beta + iu)}{Z_n(\beta)} e^{-ik_n u}\, du.$$

Taking $\beta = \beta_n(k_n)$ and using (27) this becomes

$$Q_n(k_n) = e^{-I_n(k_n) + n\ln(2d)} \times \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Z_n(\beta_n(k_n) + iu)}{Z_n(\beta_n(k_n))} e^{-ik_n u}\, du. \tag{28}$$

For the moment, K is any compact subset of (β_1^-, β_1^+). From the Taylor expansion of Z_n at $\beta = \beta_n(k_n)$ and (25), we have

$$\log Z_n(\beta_n(k_n) + iu) = \log Z_n(\beta_n(k_n)) + iuk_n - \frac{u^2}{2}\hat{D}_n + \mathrm{Rest}_n,$$

where the remainder can be estimated by the Cauchy integral formula,

$$|\mathrm{Rest}_n| \le |u|^3 \delta_K^{-3} \max\{|\log Z_n(\beta')|;\, \beta' \in D(\beta'', \delta_K), \beta'' \in K\}$$

for all $|u| \le \delta_K$, with $\delta_K > 0$ equal to half of the distance from K to the complement of U_3. From Proposition 3.1 and the definition of U_3, the above maximum is less that $C_K n$ for all $n \ge 1$, with C_K random but finite and independent of n.

Moreover, in view of Proposition 3.1 and (23,24), we see that

$$\hat{D}_n = n\lambda''(\beta_n(k_n)) + W_n''(\beta_n(k_n)) \tag{29}$$

is such that $C_K' n \le \hat{D}_n \le C_K'' n$ for some positive constants C_K', C_K''.

We split the integral in (28) according to $|u| \le \varepsilon_n := (\ln n/n)^{1/2}$ or not, and the first contribution is

$$\begin{aligned}
&\int_{|u|\le\varepsilon_n} \frac{Z_n(\beta_n(k_n)+iu)}{Z_n(\beta_n(k_n))} e^{-ik_n u}\, du \\
&\quad = \int_{|u|\le\varepsilon_n} \exp\left\{-\frac{u^2}{2}\hat{D}_n\right\} du (1+o(1)) \\
&\quad = \frac{1}{\sqrt{\hat{D}_n}} \int_{|u|\le\varepsilon_n \hat{D}_n^{1/2}} \exp\left\{-\frac{u^2}{2}\right\} du (1+o(1)) \\
&\quad = \frac{1}{\sqrt{2\pi \hat{D}_n}} (1+o(1))
\end{aligned} \tag{30}$$

since $\varepsilon_n \hat{D}_n^{1/2} \to \infty$ by (29).

Finally, to show that the other contribution is negligible, we need the following fact:

Lemma 3.3 *There exist an event Ω_3 with $\mathbb{P}(\Omega_3) = 1$, an integer random variable n_0, a neighborhood U_5 of 0 in $\mathbb{R}$, and $\kappa > 0$ such that $n_0(\eta) < \infty$ for $\eta \in \Omega_3$ and*

$$\left|\frac{Z_n(\beta+iu)}{Z_n(\beta)}\right| \le \exp\{-\kappa n u^2\} + \exp\{-\kappa n\}$$

for $\eta \in \Omega_3$, $\beta \in U_5$, $u \in [-\pi, \pi]$, and $n \ge n_0(\eta)$.

With the lemma to hand, for η, β, u as above, we bound

$$\int_{\varepsilon_n < u \le \pi} \frac{Z_n(\beta_n(k_n)+iu)}{Z_n(\beta_n(k_n))} e^{-ik_n u}\, du = o(\hat{D}_n^{-1/2})$$

where we have used $n = \mathcal{O}(\hat{D}_n)$ of (29). Combined with (30) and (28) this estimate yields the proof of the theorem, with $\Omega_2 = \Omega_1 \cap \Omega_3$, and $U_4 = U_5 \cap U_3$. □

We turn to the proof of Lemma 3.3, which states that the distribution $\nu_{n,\beta}$ does not concentrate on a sublattice of $\mathbb{Z}$, and is not too close from such a distribution. In our proof we take advantage of some (conditional) independence in the variables $\eta(t, S_t)$ under $\nu_{n,\beta}$. This is reminiscent of a construction of [11] for central limit theorem and equivalence of ensembles for Gibbs random fields.

Proof of Lemma 3.3 In the notations of the proof of Lemma 2.3,

$$\begin{aligned}
\left|\frac{Z_{2n}(\beta+iu)}{Z_{2n}(\beta)}\right| &= \left|E_{\mu_{2n}} e^{iuH_{2n}}\right| \\
&= \left|E_{\mu_{2n}} E_{\mu_{2n}}\left[e^{iuH_{2n}} \mid \Sigma^e\right]\right| \\
&\le E_{\mu_{2n}} \left|E_{\mu_{2n}}\left[e^{iuH_{2n}} \mid \Sigma^e\right]\right| \\
&= E_{\mu_{2n}} \prod_{t=1}^{n} \left|E_{\mu_{2n}}\left[e^{iu\eta(2t-1, S_{2t-1})} \mid \Sigma^e\right]\right|
\end{aligned}$$

by conditional independence of $\Sigma_1, \Sigma_3, \dots, \Sigma_{2n-1}$ under μ_{2n} given Σ^e. Recall the notation $\mathcal{I}$ from (16) and denote by

$$m_\ell = \text{Card}\{x \in \mathcal{I}(S_{2t-2}, S_{2t}) : \eta(2t-1, x) = \ell\}, \quad \ell = 0, 1, \dots,$$

the number of sites which can be reached by the walk at time $2t-1$ and where $\eta(\cdot)$ equals to 0 and 1 respectively ($m_1 + m_0 \le 2d$). Then, for $m_0, m_1 \ge 1$,

$$\begin{aligned}\left|E_{\mu_{2n}}\left[e^{iu\eta(2t+1,S_{2t+1})}\middle|\Sigma^e\right]\right| &= \left|\frac{m_1 e^{\beta+iu} + m_0}{m_1 e^\beta + m_0}\right| \\ &\le \exp\{-Cu^2\}, \quad |u| \le \pi,\end{aligned}$$

where the constant C is uniform for $\beta \in K$, and $1 \le m_0, m_1 \le 2d$. We obtain

$$\left|E_{\mu_{2n}} e^{iuH_{2n}}\right| \le E_{\mu_{2n}} \exp\left\{-Cu^2 \sum_{t=1}^n \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\}\right\}.$$

So far our arguments did not require β to be small. From this point, we will use a perturbation argument. Since $\mu_{2n}^{(\beta)}$ is equal to P for $\beta = 0$, we study the term on the right-hand side for the simple random walk measure P instead of the polymer measure μ_{2n}, and estimate the error from this change of measure. This procedure is rather weak, we believe that the result of the lemma holds for a much larger range of β, but we do not know how to control the term in the right-hand side in a different way.

For $\varepsilon > 0$ we split the last expectation according to the sum being larger or smaller than $n\varepsilon$,

$$\begin{aligned}\left|E_{\mu_{2n}} e^{iuH_{2n}}\right| &\le e^{-C\varepsilon u^2} + \mu_{2n}\left(\sum_{t=1}^n \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \le n\varepsilon\right) \\ &\le e^{-C\varepsilon u^2} + e^{2n\beta} P\left(\sum_{t=1}^n \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \le n\varepsilon\right) \qquad (31)\end{aligned}$$

by the obvious inequalities $0 \le H_{2n} \le 2n$. For $\gamma \in (0,1]$, note that

$$\begin{aligned}\mathbb{E}\exp\big\{-\gamma \mathbf{1}\{M(\eta, 2t-1, S_{2t-2}, S_{2t})\}\big\} &= e^{-\gamma} q(S_{2t-2} - S_{2t}) \\ &\quad + [1 - q(S_{2t-2} - S_{2t})],\end{aligned}$$

with q defined in (17). Then, there exists some $C_1 > 0$ such that

$$\sup_{\substack{x: P(\Sigma_2 = x) > 0, \\ \|x\|_\infty \le 1}} \big(e^{-\gamma} q(x) + [1 - q(x)]\big) \le \exp\{-C_1\gamma\}, \quad \gamma \in (0,1].$$

Hence,

$$\begin{aligned}&\mathbb{E}E_P \exp\left\{-\gamma \sum_{t=1}^n \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\}\right\} \\ &\quad = E_P \exp\left\{-C_1\gamma \sum_{t=1}^n \mathbf{1}\{\|\Sigma_{2t-2} - \Sigma_{2t}\|_\infty \le 1\}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \left(E_P \exp\left\{-C_1\gamma \mathbf{1}\{\|\Sigma_2\|_\infty \le 1\}\right\}\right)^n \\
&= \left(\frac{(2d-1)e^{-C_1\gamma}+1}{2d}\right)^n \\
&\le e^{-nC_2\gamma}
\end{aligned}
$$

with $C_2 > 0$. Now, we choose $\varepsilon = C_2/2$, $\gamma = 1$, and we get

$$
\begin{aligned}
&\mathbb{E}P\left(\sum_{t=1}^n \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \le n\varepsilon\right) \\
&\quad \le e^{n\gamma\varepsilon}\mathbb{E}E_P \exp\left\{-\gamma \sum_{t=1}^n \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\}\right\} \\
&\quad \le e^{-nC_2/2},
\end{aligned}
$$

and then

$$
\mathbb{P}\left(P\left(\sum_{t=1}^n \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \le n\varepsilon\right) \ge e^{-nC_2/4}\right) \le e^{-nC_2/4}.
$$

By Borel-Cantelli lemma, the set Ω_3 of all environments such that

$$
P\left(\sum_{t=1}^n \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \le n\varepsilon\right) \le e^{-nC_2/4} \quad \text{eventually,}
$$

is of full measure. We define n_0 as the first integer (if exists) from which the previous bound is fulfilled, and $U_5 = (-C_2/4, C_2/4)$. From (31) we easily check that Lemma 3.3 holds true with $\kappa = \min(C\varepsilon, C_2/2)$. □

Proof of Theorem 1.6 The theorem is a corollary of Theorem 3.2, where Ω_2 and U_3 are introduced. In particular we know that $\alpha = -\eta^*$ in U_3. Note that $\beta(\rho)$ is the maximizer in the definition of $\lambda^*(\rho)$ as a Legendre transform. Since $k_n/n \to \rho$, we have that $\beta_n(k_n) \to \beta(\rho)$. By (29), $\hat{D}_n \sim n\lambda''(\beta(\rho))$, and by Legendre duality,

$$
(\lambda^*)' \circ \lambda' = \mathrm{Id},
$$

and so $\lambda''(\beta(\rho)) = 1/(\lambda^*)''(\rho)$. The only quantity left to be studied is $I_n(k_n)$. Combining (26, 23) and performing the change of variable $\beta = \beta(k_n/n) + v$, we have

$$
\begin{aligned}
I_n(k_n) &= \sup\{\beta k_n - n\hat{\lambda}(\beta) - \ln W_n(\beta); \beta \in \mathbb{R}\} \\
&= \sup\big\{(\beta(k_n/n) + v)k_n - n\hat{\lambda}(\beta(k_n/n) + v) \\
&\qquad - \ln W_n(\beta(k_n/n) + v); v \in \mathbb{R}\big\} \\
&= \sup\big\{n\big[\hat{\lambda}(\beta(k_n/n)) - \hat{\lambda}(\beta(k_n/n) + v) + \hat{\lambda}'(\beta(k_n/n))v\big] \\
&\qquad - \ln W_n(\beta(k_n/n) + v); v \in \mathbb{R}\big\} + n\hat{\lambda}^*(k_n/n) \\
&= n\hat{\lambda}^*(k_n/n) - \ln W_n(\beta(k_n/n))
\end{aligned}
$$

$$
\begin{aligned}
&+\sup\big\{n\big[\hat{\lambda}(\beta(k_n/n))-\hat{\lambda}(\beta(k_n/n)+v)+\hat{\lambda}'(\beta(k_n/n))v\big]\\
&-\ln W_n(\beta(k_n/n)+v)+\ln W_n(\beta(k_n/n));\ v\in\mathbb{R}\big\}\\
=\ &n\hat{\lambda}^*(k_n/n)-\ln W_n(\beta(k_n/n))+o(1)\\
=\ &n\hat{\lambda}^*(k_n/n)-\ln W_n(\beta(\rho))+o(1)
\end{aligned}
$$

by strict convexity of $\hat{\lambda}$ and the fact that $|\ln[W_n(\beta+v)/W_n(\beta)]|\le|v|$. □

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References

1. Biggins, J.: Uniform convergence of martingales in the branching random walk. Ann. Probab. **20**, 137–151 (1992)
2. Boldrighini, C., Minlos, R.A., Pellegrinotti, A.: Directed polymers up to the L_2 threshold. Markov Processes Relat. Fields **12**, 475–508 (2006)
3. Bolthausen, E.: A note on diffusion of directed polymers in a random environment. Commun. Math. Phys. **123**, 529–534 (1989)
4. Campanino, M., Ioffe, D.: Ornstein-Zernike theory for the Bernoulli bond percolation on $\mathbb{Z}^d$. Ann. Probab. **30**, 652–682 (2002)
5. Comets, F., Shiga, T., Yoshida, N.: Directed polymers in random environment: path localization and strong disorder. Bernoulli **9**, 705–723 (2003)
6. Comets, F., Shiga, T., Yoshida, N.: Probabilistic analysis of directed polymers in a random environment: a review. In: Funaki, T., Osada, H. (eds.) Stochastic Analysis on Large Scale Interacting Systems. Advanced Studies in Pure Mathematics, vol. 39, pp. 115–142 (2004)
7. Comets, F., Vargas, V.: Majorizing multiplicative cascades for directed polymers in random media. ALEA Lat. Am. J. Probab. Math. Stat. **2**, 267–277 (2006)
8. Comets, F., Yoshida, N.: Directed polymers in random environment are diffusive at weak disorder. Ann. Probab. **34**, 1746–1770 (2006)
9. Dembo, A., Zeitouni, O.: Large Deviation Techniques and Applications, 2nd edn. Springer, New York (1998)
10. Dobrushin, R., Nahapetian, B.: Strong convexity of the pressure for lattice systems of classical statistical physics. Teor. Mat. Fiz. **20**, 223–234 (1974). (Russian). English translation: Theor. Math. Phys. **20**, 782–789 (1974)
11. Dobrushin, R., Tirozzi, B.: The central limit theorem and the problem of equivalence of ensembles. Commun. Math. Phys. **54**, 173–192 (1977)
12. Durrett, R.: Lecture Notes on Particle Systems and Percolation. Brooks Cole Statistics Probability Series. Wadsworth, Pacific Grove (1988)
13. Grimmett, G.: Percolation, 2nd edn. Fundamental Principles of Mathematical Sciences, vol. 321. Springer, Berlin (1999)
14. Imbrie, J., Spencer, T.: Diffusion of directed polymer in a random environment. J. Stat. Phys. **52**, 609–626 (1988)
15. Kesten, H.: Aspects of First Passage Percolation. École d'été de probabilités de Saint-Flour XIV. Lecture Notes in Math., vol. 1180, pp. 125–264. Springer, Berlin (1986)
16. Kesten, H., Su, Z.: Asymptotic behavior of the critical probability for ρ-percolation in high dimensions. Probab. Theory Relat. Fields **117**, 419–447 (2000)
17. Kesten, H., Sidoravicius, V.: A problem in last-passage percolation. Preprint, arXiv.org/abs/0706.3626 (2007)
18. Kurkova, I.: Local energy statistics in directed polymers. Electron. J. Probab. **13**, 5–25 (2008)
19. Lee, S.: An inequality for greedy lattice animals. Ann. Appl. Probab. **3**, 1170–1188 (1993)
20. Menshikov, M., Zuev, S.: Models of ρ-percolation. In: Kolchin, V.F., et al. (eds.) Petrozavodsk Conference on Probabilistic Methods in Discrete Mathematics. Progr. Pure Appl. Discrete Math., vol. 1, pp. 337–347. VSP, Utrecht (1993)

21. Petrov, V.: Sums of Independent Random Variables. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82. Springer, New York (1975)
22. Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987)
23. Sinai, Y.: A remark concerning random walks with random potentials. Fundom. Math. **147**, 173–180 (1995)
24. Spitzer, F.: Principles of Random Walks. Springer, Berlin (1976)